

Operadic quantization of VII_a , $\text{III}_{a=1}$, $\text{VI}_{a \neq 1}$ over harmonic oscillator

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Abstract

Operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of some real three dimensional Lie algebras. The Jacobi operators of these quantum algebras are studied.

1 Introduction and outline of the paper

In Hamiltonian formalism, a mechanical system is described by the canonical variables q^i, p_i and their time evolution is prescribed by the Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad (1.1)$$

By a Lax representation [3] of a mechanical system one means such a pair (L, M) of matrices (linear operators) L, M that the above Hamiltonian system may be represented as the Lax equation

$$\frac{dL}{dt} = ML - LM \quad (1.2)$$

Thus, from the algebraic point of view, mechanical systems may be represented by linear operators, i.e by linear maps $V \rightarrow V$ of a vector space V . As a generalization of this one can pose the following question [4]: how to describe the time evolution of the linear operations (multiplications) $V^{\otimes n} \rightarrow V$?

The algebraic operations (multiplications) can be seen as an example of the *operadic* variables [1]. If an operadic system depends on time one can speak about *operadic dynamics* [4]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of the operadic variables may be given by the operadic Lax equation. In [5, 6, 7], the low-dimensional binary operadic Lax representations for the harmonic oscillator were constructed. In [8] it was shown how the operadic Lax representations are related to the conservation of energy.

Operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of some real three dimensional Lie algebras. The Jacobi operators of these quantum algebras are studied.

2 Endomorphism operad and Gerstenhaber brackets

Let K be a unital associative commutative ring, V be a unital K -module, and $\mathcal{E}_V^n := \text{End}_V^n := \text{Hom}(V^{\otimes n}, V)$ ($n \in \mathbb{N}$). For an *operation* $f \in \mathcal{E}_V^n$, we refer to n as the *degree* of f and often write

(when it does not cause confusion) f instead of $\deg f$. For example, $(-1)^f := (-1)^n$, $\mathcal{E}_V^f := \mathcal{E}_V^n$ and $\circ_f := \circ_n$. Also, it is convenient to use the *reduced* degree $|f| := n - 1$. Throughout this paper, we assume that $\otimes := \otimes_K$.

Definition 2.1 (endomorphism operad [1]). For $f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g$ define the *partial compositions*

$$f \circ_i g := (-1)^{i|g|} f \circ (\text{id}_V^{\otimes i} \otimes g \otimes \text{id}_V^{\otimes (|f|-i)}) \in \mathcal{E}_V^{f+|g|}, \quad 0 \leq i \leq |f|$$

The sequence $\mathcal{E}_V := \{\mathcal{E}_V^n\}_{n \in \mathbb{N}}$, equipped with the partial compositions \circ_i , is called the *endomorphism operad* of V .

Definition 2.2 (total composition [1]). The *total composition* $\circ: \mathcal{E}_V^f \otimes \mathcal{E}_V^g \rightarrow \mathcal{E}_V^{f+|g|}$ is defined by

$$f \circ g := \sum_{i=0}^{|f|} f \circ_i g \in \mathcal{E}_V^{f+|g|}, \quad |\circ| = 0$$

The pair $\text{Com } \mathcal{E}_V := \{\mathcal{E}_V, \circ\}$ is called the *composition algebra* of \mathcal{E}_V .

Definition 2.3 (Gerstenhaber brackets [1]). The *Gerstenhaber brackets* $[\cdot, \cdot]$ are defined in $\text{Com } \mathcal{E}_V$ as a graded commutator by

$$[f, g] := f \circ g - (-1)^{|f||g|} g \circ f = -(-1)^{|f||g|} [g, f], \quad |[\cdot, \cdot]| = 0$$

The *commutator algebra* of $\text{Com } \mathcal{E}_V$ is denoted as $\text{Com}^- \mathcal{E}_V := \{\mathcal{E}_V, [\cdot, \cdot]\}$. One can prove (e.g [1]) that $\text{Com}^- \mathcal{E}_V$ is a *graded Lie algebra*. The Jacobi identity reads

$$(-1)^{|f||h|} [f, [g, h]] + (-1)^{|g||f|} [g, [h, f]] + (-1)^{|h||g|} [h, [f, g]] = 0$$

3 Operadic dynamics and Lax equation

Assume that $K := \mathbb{R}$ or $K := \mathbb{C}$ and operations are differentiable. Dynamics in operadic systems (operadic dynamics) may be introduced by

Definition 3.1 (operadic Lax pair [4]). Allow a classical dynamical system to be described by the Hamiltonian system (1.1). An *operadic Lax pair* is a pair (μ, M) of homogeneous operations $\mu, M \in \mathcal{E}_V$, such that the Hamiltonian system (1.1) may be represented as the *operadic Lax equation*

$$\frac{d\mu}{dt} = [M, \mu] := M \circ \mu - (-1)^{|M||\mu|} \mu \circ M$$

The pair (L, M) is also called an *operadic Lax representations* of/for Hamiltonian system (1.1). Evidently, the degree constraints $|M| = |L| = 0$ give rise to ordinary Lax equation (1.2) [3]. In this paper we assume that $|M| = 0$.

The Hamiltonian of the harmonic oscillator (HO) is

$$H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

Thus, the Hamiltonian system of HO reads

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q \quad (3.1)$$

If μ is a linear algebraic operation we can use the above Hamilton equations to obtain

$$\frac{d\mu}{dt} = \frac{\partial\mu}{\partial q} \frac{dq}{dt} + \frac{\partial\mu}{\partial p} \frac{dp}{dt} = p \frac{\partial\mu}{\partial q} - \omega^2 q \frac{\partial\mu}{\partial p} = [M, \mu]$$

Therefore, we get the following linear partial differential equation for $\mu(q, p)$:

$$p \frac{\partial\mu}{\partial q} - \omega^2 q \frac{\partial\mu}{\partial p} = [M, \mu] \quad (3.2)$$

By integrating (3.2) one can get collections of operations called [4] the *operadic* (Lax representations for/of) harmonic oscillator.

4 3D binary anti-commutative operadic Lax representations for harmonic oscillator

Lemma 4.1. *Matrices*

$$L := \begin{pmatrix} p & \omega q & 0 \\ \omega q & -p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M := \frac{\omega}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

give a 3-dimensional Lax representation for the harmonic oscillator.

Definition 4.2 (quasi-canonical coordinates). For the HO, define its *quasi-canonical coordinates* Q and P by

$$P^2 - Q^2 = 2p, \quad QP = \omega q \quad (4.1)$$

Remark 4.3. Note that these constraints easily imply

$$P^2 + Q^2 = 2\sqrt{2H}$$

Theorem 4.4 ([7]). Let $C_\nu \in \mathbb{R}$ ($\nu = 1, \dots, 9$) be arbitrary real-valued parameters, such that

$$C_2^2 + C_3^2 + C_5^2 + C_6^2 + C_7^2 + C_8^2 \neq 0 \quad (4.2)$$

Let M be defined as in Lemma 4.1 and $\mu : V \otimes V \rightarrow V$ be a binary operation in a 3 dimensional real vector space V with the coordinates

$$\left\{ \begin{array}{l} \mu_{11}^1 = \mu_{22}^1 = \mu_{33}^1 = \mu_{11}^2 = \mu_{22}^2 = \mu_{33}^2 = \mu_{11}^3 = \mu_{22}^3 = \mu_{33}^3 = 0 \\ \mu_{23}^1 = -\mu_{32}^1 = C_2 p - C_3 \omega q - C_4 \\ \mu_{13}^2 = -\mu_{31}^2 = C_2 p - C_3 \omega q + C_4 \\ \mu_{31}^1 = -\mu_{13}^1 = C_2 \omega q + C_3 p - C_1 \\ \mu_{23}^2 = -\mu_{32}^2 = C_2 \omega q + C_3 p + C_1 \\ \mu_{12}^1 = -\mu_{21}^1 = C_5 P + C_6 Q \\ \mu_{12}^2 = -\mu_{21}^2 = C_5 Q - C_6 P \\ \mu_{13}^3 = -\mu_{31}^3 = C_7 P + C_8 Q \\ \mu_{23}^3 = -\mu_{32}^3 = C_7 Q - C_8 P \\ \mu_{12}^3 = -\mu_{21}^3 = C_9 \end{array} \right. \quad (4.3)$$

Then (μ, M) is an operadic Lax pair for HO.

5 Initial conditions

Specify the coefficients C_ν in Theorem 4.4 by the initial conditions

$$\mu|_{t=0} = \overset{\circ}{\mu}, \quad p|_{t=0} = p_0, \quad q|_{t=0} = 0$$

Denoting $E := H|_{t=0}$, the latter together with (4.1) yield the initial conditions for Q and P :

$$\begin{cases} (P^2 + Q^2)|_{t=0} = 2\sqrt{2E} \\ (P^2 - Q^2)|_{t=0} = 2p_0 \\ PQ|_{t=0} = 0 \end{cases} \iff \begin{cases} p_0 > 0 \\ P^2|_{t=0} = 2p_0 \\ Q|_{t=0} = 0 \end{cases} \vee \begin{cases} p_0 < 0 \\ P|_{t=0} = 0 \\ Q^2|_{t=0} = -2p_0 \end{cases}$$

In what follows assume that $p_0 > 0$ and $P|_{t=0} = \sqrt{2p_0}$. The other cases can be treated similarly. Note that in this case $p_0 = \sqrt{2E}$. From (4.3) we get the following linear system:

$$\begin{cases} C_1 = \frac{1}{2} \left(\overset{\circ}{\mu}_{23}^2 - \overset{\circ}{\mu}_{31}^1 \right), & C_2 = \frac{1}{2p_0} \left(\overset{\circ}{\mu}_{13}^2 + \overset{\circ}{\mu}_{23}^1 \right), & C_3 = \frac{1}{2p_0} \left(\overset{\circ}{\mu}_{23}^2 + \overset{\circ}{\mu}_{31}^1 \right) \\ C_4 = \frac{1}{2} \left(\overset{\circ}{\mu}_{13}^2 - \overset{\circ}{\mu}_{23}^1 \right), & C_5 = \frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{12}^1, & C_6 = -\frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{12}^2 \\ C_7 = \frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{13}^3, & C_8 = -\frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{23}^3, & C_9 = \overset{\circ}{\mu}_{12}^3 \end{cases} \quad (5.1)$$

6 VII_a, III_{a=1}, VI_{a≠1}

We study only the algebras VII_a, III_{a=1}, VI_{a≠1} from the Bianchi classification of the real three dimensional Lie algebras [2]. The structure equations of the 3-dimensional real Lie algebras can be presented as follows:

$$[e_1, e_2] = -\alpha e_2 + n^3 e_3, \quad [e_2, e_3] = n^1 e_1, \quad [e_3, e_1] = n^2 e_2 + \alpha e_3$$

The values of the parameters α, n^1, n^2, n^3 and the corresponding structure constants for II, VII_a, III_{a=1}, VI_{a≠1} are presented in Table 6.1. Note that II is the real three dimensional Heisenberg algebra.

Bianchi type	α	n^1	n^2	n^3	$\overset{\circ}{\mu}_{12}^1$	$\overset{\circ}{\mu}_{12}^2$	$\overset{\circ}{\mu}_{12}^3$	$\overset{\circ}{\mu}_{23}^1$	$\overset{\circ}{\mu}_{23}^2$	$\overset{\circ}{\mu}_{23}^3$	$\overset{\circ}{\mu}_{31}^1$	$\overset{\circ}{\mu}_{31}^2$	$\overset{\circ}{\mu}_{31}^3$
VII _a	a	0	1	1	0	$-a$	1	0	0	0	0	1	a
III _{a=1}	1	0	1	-1	0	-1	-1	0	0	0	0	1	1
VI _{a≠1}	a	0	1	-1	0	$-a$	-1	0	0	0	0	1	a

Table 6.1: VII_a, III_{a=1}, VI_{a≠1}. Here $a > 0$.

7 VII_a^t, III_{a=1}^t, VI_{a≠1}^t

By using the structure constants of the 3-dimensional Lie algebras in the Bianchi classification, Theorem 4.4 and relations (5.1) one can propose that evolution of VII_a, III_{a=1}, VI_{a≠1} can be prescribed [8] as given in Table 7.1.

Dynamical Bianchi type	μ_{12}^1	μ_{12}^2	μ_{12}^3	μ_{23}^1	μ_{23}^2	μ_{23}^3	μ_{31}^1	μ_{31}^2	μ_{31}^3
VII_a^t	$\frac{aQ}{\sqrt{2p_0}}$	$\frac{-aP}{\sqrt{2p_0}}$	1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aQ}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aP}{\sqrt{2p_0}}$
$\text{III}_{a=1}^t$	$\frac{Q}{\sqrt{2p_0}}$	$\frac{-P}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-Q}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{P}{\sqrt{2p_0}}$
$\text{VI}_{a \neq 1}^t$	$\frac{aQ}{\sqrt{2p_0}}$	$\frac{-aP}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aQ}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aP}{\sqrt{2p_0}}$

Table 7.1: $\text{VII}_a^t, \text{III}_{a=1}^t, \text{VI}_{a \neq 1}^t$. Here $p_0 = \sqrt{2E}$.

8 $\text{VII}_a^h, \text{III}_{a=1}^h, \text{VI}_{a \neq 1}^h$ and quantum Jacobi operators

By using the algebras $\text{VII}_a^t, \text{III}_{a=1}^t, \text{VI}_{a \neq 1}^t$ from Table 7.1, one can propose [9] their quantum counterparts $\text{VII}_a^h, \text{III}_{a=1}^h, \text{VI}_{a \neq 1}^h$ as follows.

Let \mathcal{A}_{HO} denote the state space of the quantum harmonic oscillator and $\{e_1, e_2, \dots\}$ be its basis. By using Table 8.1 we define the structure equations in \mathcal{A}_{HO} by

$$[e_i, e_j]_h := \hat{\mu}_{ij}^s e_s$$

where the structure operators $\hat{\mu}_{ij}^s$ for $i, j, s \leq 3$ are defined by Table 8.1 and $\hat{\mu}_{ij}^s := 0$ for $i, j, s > 3$. For $x, y \in \mathcal{A}_{HO}$, their quantum multiplication is defined by

Quantum Bianchi type	$\hat{\mu}_{12}^1$	$\hat{\mu}_{12}^2$	$\hat{\mu}_{12}^3$	$\hat{\mu}_{23}^1$	$\hat{\mu}_{23}^2$	$\hat{\mu}_{23}^3$	$\hat{\mu}_{31}^1$	$\hat{\mu}_{31}^2$	$\hat{\mu}_{31}^3$
VII_a^h	$\frac{a\hat{Q}}{\sqrt{2p_0}}$	$\frac{-a\hat{P}}{\sqrt{2p_0}}$	1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{Q}}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{P}}{\sqrt{2p_0}}$
$\text{III}_{a=1}^h$	$\frac{\hat{Q}}{\sqrt{2p_0}}$	$\frac{-\hat{P}}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-\hat{Q}}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{\hat{P}}{\sqrt{2p_0}}$
$\text{VI}_{a \neq 1}^h$	$\frac{a\hat{Q}}{\sqrt{2p_0}}$	$\frac{-a\hat{P}}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{Q}}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{P}}{\sqrt{2p_0}}$

Table 8.1: $\text{VII}_a^h, \text{III}_{a=1}^h, \text{VI}_{a \neq 1}^h$.

$$[x, y]_h := \hat{\mu}_{jk}^i x^j y^k e_i = \hat{\mu}_{jk}^1 x^j y^k e_1 + \hat{\mu}_{jk}^2 x^j y^k e_2 + \hat{\mu}_{jk}^3 x^j y^k e_3$$

where we missed the trivial terms, because $\hat{\mu}_{jk}^i = 0$ for $i > 3$. Then the quantum Jacobi operator is defined by

$$\begin{aligned} \hat{J}_h(x; y; z) &:= [x, [y, z]_h]_h + [y, [z, x]_h]_h + [z, [x, y]_h]_h \\ &= \hat{J}_h^1(x; y; z) e_1 + \hat{J}_h^2(x; y; z) e_2 + \hat{J}_h^3(x; y; z) e_3 \end{aligned}$$

where we again missed the trivial terms, because $\hat{J}_h^i = 0$ for $i > 3$. In [9] the quantum Jacobi operators were calculated for all real three dimensional Lie algebras. Here we concentrate only

on $\text{III}_{a=1}^{\hbar}$, $\text{VI}_{a \neq 1}^{\hbar}$, and VII_a^{\hbar} . Denote

$$(x, y, z) := \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}, \quad \hat{\xi}^1 := \omega \hat{q} \hat{Q} + (\hat{p} - p_0) \hat{P}, \quad \hat{\xi}^2 := \omega \hat{q} \hat{P} - (\hat{p} + p_0) \hat{Q}$$

Recall

Theorem 8.1 ([9]). *The Jacobi operator components of $\text{VI}_{a \neq 1}^{\hbar}$ and VII_a^{\hbar} read*

$$\hat{J}_h^1(x; y; z) = -\frac{a(x, y, z)}{\sqrt{2p_0^3}} \hat{\xi}^1, \quad \hat{J}_h^2(x; y; z) = -\frac{a(x, y, z)}{\sqrt{2p_0^3}} \hat{\xi}^2, \quad \hat{J}_h^3(x; y; z) = \frac{a^2(x, y, z)}{p_0} [\hat{P}, \hat{Q}]$$

For $\text{III}_{a=1}^{\hbar}$ one has the same formulae with $a = 1$.

9 Quasi-canonical quantum conditions

Theorem 9.1 (Poisson brackets of quasi-canonical coordinates). *The quasi-canonical coordinates Q and P satisfy the relations*

$$\{P, P\} = 0 = \{Q, Q\}, \quad \{P, Q\} = \varepsilon := \frac{\omega}{2\sqrt{2H}} \quad (9.1)$$

Proof. While the first two relations in (9.1) are evident, we have only to check the third one. Using several times the Leibniz rule for the Poisson brackets, calculate:

$$\begin{aligned} 2\omega &= 2\omega\{p, q\} = \{P^2 - Q^2, PQ\} \\ &= \{P^2, PQ\} - \{Q^2, PQ\} \\ &= P\{P^2, Q\} - \{Q^2, P\}Q \\ &= P\{PP, Q\} - \{QQ, P\}Q \\ &= 2(P^2 + Q^2)\{P, Q\} \\ &= 4\sqrt{2H}\{P, Q\} \end{aligned}$$

□

When performing the quantization of the quasi-canonical variables, we shall use the Schrödinger picture, i.e the operators $\hat{q}, \hat{p}, \hat{H}$ and \hat{Q}, \hat{P} do not depend on time. Denote by $[\cdot, \cdot]$ the ordinary commutator bracketing. Following the canonical quantization prescription, the quasi-canonical coordinates would satisfy the constraints

$$\hat{P}^2 + \hat{Q}^2 = 2\sqrt{2\hat{H}}, \quad \hat{P}^2 - \hat{Q}^2 = 2\hat{p}, \quad \hat{P}\hat{Q} + \hat{Q}\hat{P} = 2\omega\hat{q} \quad (9.2)$$

and the *quasi-canonical commutation relations* (quasi-CCR) read as follows:

$$[\hat{P}, \hat{P}] = 0 = [\hat{Q}, \hat{Q}], \quad [\hat{P}, \hat{Q}] = \frac{\hbar}{i} \hat{\varepsilon} := \frac{\hbar}{i} \frac{\omega}{2\sqrt{2\hat{H}}} \quad (9.3)$$

10 Recapitulation

Theorem 10.1. *Let constraints (9.2) and (9.3) hold. Then we have:*

$$\begin{aligned} \hat{J}_h^1(x; y; z) &= \frac{a(x, y, z)}{\sqrt{2p_0^3}} \left[\hat{P} \left(\sqrt{2E} - \sqrt{2\hat{H}} \right) - \frac{\hbar}{i} \hat{Q} \frac{\hat{\varepsilon}}{2} \right] \\ \hat{J}_h^2(x; y; z) &= \frac{a(x, y, z)}{\sqrt{2p_0^3}} \left[\hat{Q} \left(\sqrt{2E} - \sqrt{2\hat{H}} \right) + \frac{\hbar}{i} \hat{P} \frac{\hat{\varepsilon}}{2} \right] \\ \hat{J}_h^3(x; y; z) &= \frac{\hbar}{i} \frac{a^2(x, y, z)}{p_0} \hat{\varepsilon} \end{aligned}$$

□

Proof. Using relations (9.2) and (9.3) first calculate:

$$\begin{aligned} \hat{\xi}^1 &= \omega \hat{q} \hat{Q} + (\hat{p} - p_0) \hat{P} \\ &= \frac{1}{2} (\hat{P} \hat{Q} + \hat{Q} \hat{P}) \hat{Q} + \frac{1}{2} (\hat{P}^2 - \hat{Q}^2) \hat{P} - p_0 \hat{P} \\ &= \frac{1}{2} (\hat{P} \hat{Q}^2 + \hat{Q} \hat{P} \hat{Q} + \hat{P}^3 - \hat{Q}^2 \hat{P}) - p_0 \hat{P} \\ &= \frac{1}{2} \left[\hat{Q} (\hat{P} \hat{Q} - \hat{Q} \hat{P}) + \hat{P} (\hat{Q}^2 + \hat{P}^2) \right] - p_0 \hat{P} \\ &= \frac{1}{2} \hat{Q} [\hat{P}, \hat{Q}] + \frac{1}{2} \hat{P} (\hat{P}^2 + \hat{P}^2) - p_0 \hat{P} \\ &= \frac{\hbar}{i} \hat{Q} \frac{\hat{\varepsilon}}{2} + \hat{P} \sqrt{2\hat{H}} - \sqrt{2E} \hat{P} \\ &= \frac{\hbar}{i} \hat{Q} \frac{\hat{\varepsilon}}{2} + \hat{P} \left(\sqrt{2\hat{H}} - \sqrt{2E} \right) \end{aligned}$$

Next calculate

$$\begin{aligned} \hat{\xi}^2 &= \omega \hat{q} \hat{P} - (\hat{p} + p_0) \hat{Q} \\ &= \frac{1}{2} (\hat{P} \hat{Q} + \hat{Q} \hat{P}) \hat{P} - \frac{1}{2} (\hat{P}^2 - \hat{Q}^2) \hat{Q} - p_0 \hat{Q} \\ &= \frac{1}{2} (\hat{P} \hat{Q} \hat{P} + \hat{Q} \hat{P}^2 - \hat{P}^2 \hat{Q} + \hat{Q}^3) - p_0 \hat{Q} \\ &= \frac{1}{2} \left[\hat{P} (\hat{Q} \hat{P} - \hat{P} \hat{Q}) + \hat{Q} (\hat{P}^2 + \hat{Q}^2) \right] - p_0 \hat{Q} \\ &= -\frac{1}{2} \hat{P} [\hat{P}, \hat{Q}] + \frac{1}{2} \hat{Q} (\hat{P}^2 + \hat{Q}^2) - p_0 \hat{Q} \\ &= -\frac{\hbar}{i} \hat{P} \frac{\hat{\varepsilon}}{2} + \hat{Q} \sqrt{2\hat{H}} - \sqrt{2E} \hat{Q} \\ &= -\frac{\hbar}{i} \hat{P} \frac{\hat{\varepsilon}}{2} + \hat{Q} \left(\sqrt{2\hat{H}} - \sqrt{2E} \right) \end{aligned}$$

□

Corollary 10.2. *Let constraints (9.2), (9.3) and $\hat{H} = E$ hold. Then we have*

$$\begin{aligned}\hat{J}_h^1(x; y; z) &= -\frac{\hbar a(x, y, z)}{i} \frac{\omega}{\sqrt{(2p_0)^3}} \frac{1}{2\sqrt{2E}} \hat{Q} \\ \hat{J}_h^2(x; y; z) &= +\frac{\hbar a(x, y, z)}{i} \frac{\omega}{\sqrt{(2p_0)^3}} \frac{1}{2\sqrt{2E}} \hat{P} \\ \hat{J}_h^3(x; y; z) &= +\frac{\hbar a^2(x, y, z)}{i} \frac{\omega}{p_0} \frac{1}{2\sqrt{2E}}\end{aligned}$$

□

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